RECOVERING MATHEMATICAL AWARENESS BY LINGUISTIC ANALYSIS OF VARIABLE SUBSTITUTION

Kaenders[a], R.H., Kvasz[b], L., Weiss-Pidstrygach[c],Y.

University of Cologne[a], Charles University in Prague[b], University of Mainz[c]

How are we to understand and distinguish qualities of mathematical acquirement? The authors show with help of a school relevant example how mathematical awareness can be classified. The classification uses linguistic methods to identify factors important for conceptual understanding of mathematics. The analysis relates historical to activity theoretical aspects.

INTRODUCTION

Usually mathematical awareness of a student puts itself into effect in his approach to problems, a mathematical topic, in the way the student conceptualizes a mathematical object and uses it as tool. We use this aspect of student`s cognition in order to relate two models describing the quality of mathematical acquirement from different perspectives both by linguistic means.

The first model takes an activity-theory approach (Weiss-Pidstrygach, 2011). Symbols, definitions, formulas, skills and approaches are linked to object-oriented activities, externalisation and internalisation. The second model uses different qualities of ***mathematical awareness*** (Kaenders & Kvasz, 2011, 2010) to describe mathematical aptitude. The classification is given by the three dimensions (content, skill and thinking activity) and by the quality these dimensions are linked together. The classification by different qualities of awareness is based on historical considerations and patterns of change in the development of mathematical language (Kvasz, 2008).

This approach is based on three basic convictions:

- Mathematical awareness is a *holistic* concept that unites such qualities like number sense in arithmetic, symbol sense in algebra and geometrical awareness in geometry.
- It is *topically neutral*, i.e. awareness acquired in one area can be transferred to another.
- It has different degrees that are closely related to *different degrees of rigor*.

In order to motivate a discussion about possible combining frames of these two models, their further differentiation, their relevance for mathematics education we explain our approaches with the help of an example from the school curriculum.

VARIABLE SUBSTITUTION AT SCHOOL

How do we have to modify the graph of f in order to find the graph of the function g which is given by $g(x)=f(x+3)$?

The problem is standard and appears first for a quadratic function f in Grade 8 and in the above general phrasing in the context of modeling and preparation of the "small chain rule" in Grade 10. (German curriculum)

Understanding of the mathematical question takes for granted some experience with the mathematical objects involved in the formulation of the problem as well as their relations and role in a wider context. Examples of prerequisite mathematical objects in the German curriculum are coordinate system, units, correspondence, linear functions, different representations of functional dependencies with and without calculator (value table, graph, term, text), standard functions, such as rational, trigonometric, exponential functions. The understanding and handling of these objects by students depends a lot on the attitude of the teacher and ranges from recalling definitions or knowing the calculator commands up to their conceptual use as a tool.

In the last decade, the theoretical approach to functional dependencies in textbooks became extremely pictorial, bounded to CAS supported graphs and concrete functions. On the other hand, later on, in the context of modeling, maximum and minimum value problems and integration problems are often formulated in algebraic notations (e.g. $f(x)=\sin(ax+b)$).

The existing teaching and textbook culture defines a certain linguistic frame for the determination of the graph in question.

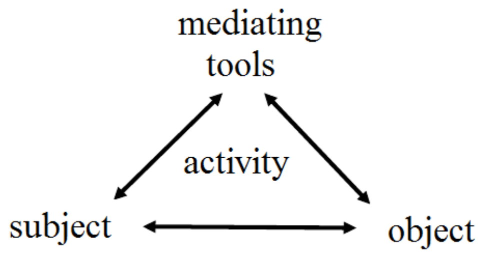
The Graph of $g(x)=f(x+3)$ can be found by various thinking activities, using various skills, solving problems formulated in various contexts and answering differently interpreted questions, following instructions or working in a group, trying to solve a problem or to teach the rules to somebody else...

The solution itself tells little about the mathematical acquaintance of the student, it could e.g. be an imitation learned by rote. His mathematical awareness manifests itself in the way the student talks about his solution, his first intuition, the appearance and development of his assumptions, doubts he has, thought experiments he went through. Thereby his possibilities to couch generalizations, analogies, counter examples etc. will depend on the language of the mathematical concepts of which the objects involved make a part.

In different concepts the denotation given to the objects can coincide: the (pointwise defined) ***polynomial function*** is also the ***polynomial function*** defined by its coefficients and is also the ***polynomial function*** as an element of the set of all continuous functions.

The meaning of the mathematical object as a concept itself is constituted by different contexts the object can be embedded into (part of existing mathematical theories), as well as by the problems which can be solved by using it as a tool. The tool-object-duality was explicitly introduced by R. Douady and used in the context of didactical engineering for concept development (e.g. Douady, 1997). We interpret this principle in an activity frame in order to change the constructivist perspective by a social-historical one.

Speaking in the terminology of activity theory: the mathematical concept "substitution of variables" can in the model of an action appear in two positions: as an object and as a mediating tool. The general model used in activity theory can be represented by the diagram on the right.



This general scheme can be applied to a concrete mathematical problem in two different ways (see fig. 1).

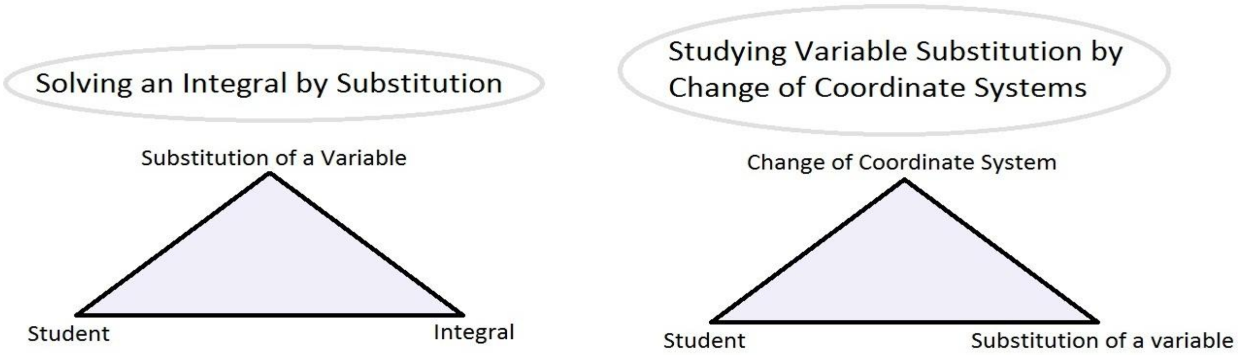


Figure 1: Substitution of variables as an object and as a mediating tool.

We can for instance consider variable substitution as a special case of a considerable more general method (concept): structure preserving transformations. Typical problems, which are solved by using structure preserving transformations are: finding a representative with the same structure but easier to handle (e.g. nicer coordinates) or transforming a given object in order to get a whole class with the same properties (possible solutions of an equation). We can also look at variable substitution as special case of the method of introducing coordinates, i.e. coordinatization. Typical problems handled by this method are finding explicit solutions in local coordinates (Taylor expansions, equations on manifolds…)

In other words the given mathematical object can be conceptualized in different ways, creating different methods for problem solving. Also the mathematical object on its own can be defined in different ways using different mathematical languages – and therefore predetermining different conceptual developments.

In the following we sketch different approaches to find the graph of $g(x)=f(x+3)$ and indicate the concept it belongs to. To this end we represent the respective correspondence given by value table, graph or term. The students are used to situations in arithmetic where '+3' indicates '3 to the right' or '3 upwards'. For the graph of $g(x)=f(x)+3$ this rule of thumb seems to be consistent. Finding the graph of $g(x)=f(x+3)$ therefore causes for many students an intuitive conflict.

POSSIBLE APPROACHES TO FINDING THE GRAPH OF $f(x+3)$

The problem of finding the graph of $g(x)=f(x+3)$ has a rich mathematical and didactic structure. There are many different approaches which the teacher can adopt. We would like to mention ten ***examples***:

54 Given a function f by $x \mapsto f(x)$, take the definition of g and evaluate for any x the function f at x+3. Draw the graph of g from the table of values.

55 Insert the term of a special function f in a graphical calculator, compute and plot the associated value table of the function $g(x)=f(x+3)$.

56 We can use path-time diagrams as a metaphor for a general function. For instance think of two motorcyclists driving exactly the same way. The path-time diagram of one of them is given by the graph of f i.e. at time x she is at distance $f(x)$. Her colleague however has started already 3 minutes earlier and has at time x already reached $f(x + 3)$. Of course, the path-time diagram of the early rider has to be drawn three units to the left from the one who departs on time.

57 We can also interpret the formula $g(x)=f(x+3)$ as recipe for *picking up values* of g (see Fig. 2). If we want to determine the height of the graph of g at the value x we go 3 units to the right and pick up the value of f. When we proceed like this several times it becomes apparent that the graph of f will be shifted 3 units to the left.

58 Another way to look at this particular shift to the left is to use nomograms for the composition with the linear function $x+3$ and to combine it with the graph of f (see Fig.2).

Substitution of the variable using the set-theoretic definition of the graph: We can consider the graph of a function h as consisting of points with coordinates of the form $(x,h(x))$. When we move the points $(x,f(x))$ of the graph of f with 3 units to the left, then a point $(x,f(x))$ is moved to $(x-3, f(x))$. Now we look for a function g such that $(x-3, f(x))$ is a point $(x', g(x'))$ on the graph of g. Hence $x' = x-3$ and $g(x')=f(x'+3)$.

These approaches consider the graph of a function as a pointwise defined object.

We can also let f be the graph of a function, perceived as a curve, drawn or plotted in a coordinate system, representing a function as an entire object.

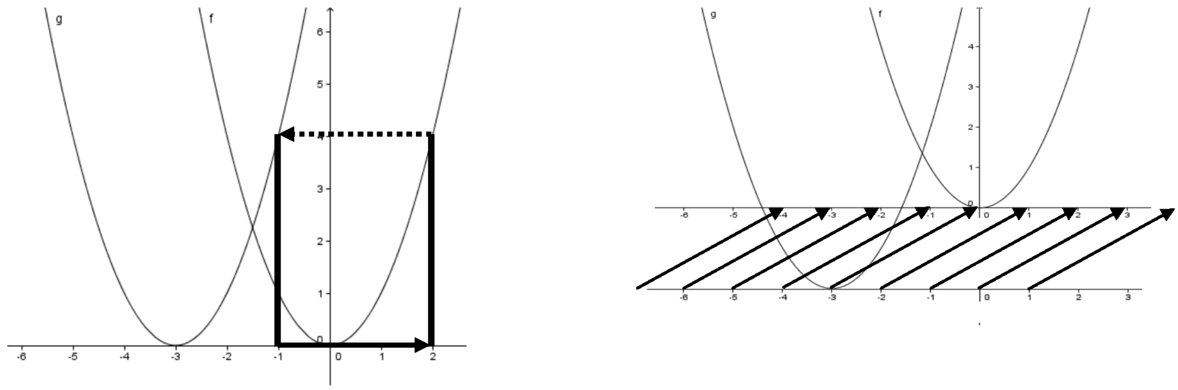


Figure 2: 'Picking up' values of g and combination of g with a nomogram.

In this situation, examples to find the graph of $g(x) = f(x+3)$ are the following.

59 Using a DGS device for shifting the curve around (e.g. GeoGebra or appropriate applets). Insert the term of a special function f and plot the corresponding graph. Shifting the graph changes the term and vice versa. It turns out that a movement to the left corresponds to adding a positive constant to the argument x of the function f .

60 By variation of the graph by e.g. dynamical geometry, using a parameter a one can see how the graph of a function $g(x) = f(x+a)$ behaves. This general observation can be confined to $a = 3$.

61 The shift $x \mapsto x+3$ can be seen as a shift of the coordinate system where the origin is moved 3 units to the right. The graph of the function f remains unchanged in the plane and hence moves – seen from the coordinate system – to the left.

62 Any function depending on a variable x can be *developed around* $x_0 = -3$. That is to find a function $g(x')$ with $g(x') = f(x'+x_0)$. The approved way to do this is to substitute x by $x'+x_0$ or to write $f(x) = f((x-x_0)+x_0)$.

The described interrelation between definition, embedding, conceptualization and operationalization of mathematical structures shows that parts of the zone of proximal development for a mathematical activity are defined and can be understood in linguistic terms. The language in which the relevant mathematical objects are named and presented provides a presetting for the local scaffolding, in particular the language for diagnostics of the zone of actual development and variations of the task. The compatibility and transferability as a tool for problem solving depends on the grammar of the concept (relations, hierarchy, dependencies between objects inside the local theory) and possible changes of perspective for the speaker (existence of a paradigm). It seems to us that these linguistic data structure the zone of proximal development in the way in which speech and motivation are interacting with thinking.

LINGUISTIC ANALYSIS OF THE EXAMPLE $f(x+3)$

For the conceptual framework of *mathematical awareness* we distinguish three aspects *contents*, *thinking* and *skills* as three main dimensions in which mathematical aptitude can be positioned. The basic idea is: the quality of the respective mathematical awareness is not an additional dimension in this diagram but it qualifies the way in which the contents, skills, and thinking are connected, thus it qualifies *the way in which* for example someone *argues* in *arithmetic* by *visualizing* or someone *proves* in *calculus* by *algebraizing*.

On the level of contents we distinguish arithmetic, synthetic geometry, algebra, analytic geometry, calculus, logic, set theory, probability.

Introduction of other subjects would be possible. The ordering of the topics follows roughly the order of historical development of mathematics, and also the growing complexity of the mathematical language (cf. Kvasz, 2008).

The dimension of skills is separated from the dimension of the thinking activities. We address the following skills: to count, to calculate, to draw, to construct, to symbolize, to algorithmize, to visualize, to recognize patterns, to establish dependencies, to use limit transitions and to employ language. Particular skills are best acquired in corresponding contents, as for instance counting in arithmetic, drawing in synthetic geometry, etc. Nevertheless, one of the reasons for separating skills from contents is to emphasize the possibility (and actually the necessity) of transference of skills from one content to another. Thus we usually apply addition to numbers, but we can add intervals, polynomials, vectors, functions, etc. So mathematical awareness is the awareness of the transferability of skills from one content to another. It is precisely the failure of many forms of mathematical instruction that particular skills are strictly tight to their corresponding content. Competence models, on the other hand, are tight to certain practical requests. Their solution requires different skills but remains on a certain level of thinking. In our model the relations between contents, skills and thinking allow a great variety of combinations.

We distinguish the following possibly not exhaustive list of thinking activities: to observe, to formulate, to argue, to explain, to verbalize, to classify, to define, to prove, to confine, to generalize, to vary, to concretize, to analogize, to structure. As in the previous case, also in the case of thinking, the role of an independent dimension emphasizes the manifold relations of thinking activities with the various topics and with the different skills. The student needs to master sufficiently a particular skill in order to be able to explain, define, or generalize a mathematical phenomenon. One of the important aspects of the mathematical awareness is the awareness of the adequate degree of rigor and justification that is sufficient for a particular calculation or construction.

Mathematical awareness then is a quality, how our knowledge of some mathematics contents, skills and thinking are linked together. It is not an extensive notion but describes intensity of the amalgam, which links these three aspects with each other. It is rather a color or a taste than a position of some mathematical ability. The list of the different qualities is possibly not exhaustive:

- social awareness
- imitative awareness
- manipulative awareness
- instrumental awareness
- diagrammatic awareness
- intuitive awareness
- experimental awareness
- strategic awareness
- contextual awareness
- argumentative awareness
- logical awareness
- theoretical awareness

Here we want to discuss the possible different linguistic indications to varying qualities of mathematical awareness that we can think of when we consider the different ways to find the graph of $g(x)= f(x+3)$.

i. Social awareness

This is the first possible level of awareness and it might be that many people stay on that level all their life. A solution to the above problem is that '+3' in $f(x+3)$ has to be interpreted as '3 units to the left' since the teacher did so in class. Moreover, also the fellow students do it that way.

- **Imitative awareness**

The above insight can also be explained by the teacher step for step. It is possible to reproduce each step of the argumentation without understanding its complete strand. And, look, it works. Also this very classical type of awareness is of great importance for the understanding of mathematics. To any student this approach gives a possibility to betake oneself in the middle of the subject guided by the authority of the teacher.

- **Manipulative awareness**

If we look at the substitution of the variable using the set-theoretic definition of the graph in ***example 6*** and the development of the function around $x_0 = -3$ in approach i), then we see that still an insight in the manipulations with variables is necessary. The procedure can be understood just from the manipulations with $x = x' + x_0$ and $f(x)= f((x-x_0)+x_0)$. In particular with polynomials we can use long division. The solution is obtained by applying these two mechanical rules in the algebraic expressions.

- **Instrumental awareness**

Plotting of a value table or using a DGS device for shifting the curve around yields instrumental awareness. It is true, that the students know the result only thanks to the use of the instrument, but the teacher would compromise his credibility if he would insist that they don't know the solution yet. Of most things in life we neither have a better knowledge nor a higher form of awareness. The instrumental awareness is what the CAS–system or likewise a graphical device tells us: the result is correct – it came out from the computer and I do not know how – not more but also not less.

- **Diagrammatic awareness**

Using $g(x)= f(x+3)$ as recipe for *picking up values* in ***example*** d) or the interpretation by a nomogram in ***example*** e) as well as the shift $x \mapsto x+3$ of the coordinate system in ***example*** i) are ways to come to a diagrammatic awareness. In any branch of mathematics pictures, graphs and diagrams play a central role. We can represent for example, arithmetic relations by dot pictures, functions by graphs or nomograms, polynomials by Newton diagrams. It is possible to argue, to define and to perform most thinking activities in a diagrammatic way. Diagrammatic awareness is indispensible for understanding mathematics, although there is always a risk by stopping at the level of diagrammatic awareness to prevent further growth to logical and theoretical awareness.

- **Intuitive awareness**

The expectation, based on experiences in arithmetic, that '+3' indicates '3 to the right' or '3 upwards' is an example of a wrong intuitive awareness. The shift of the coordinate system to the right in ***example*** i) or the adoption of a nomogram in ***example*** e) on the other hand are able to foster a better intuitive awareness. In some cases there exists a feeling, which leads to a hypothesis. That is different from experimental awareness since there we only find an assertion to be true in the case of some cases.

- **Experimental awareness**

The calculating and plotting of a value table in ***example*** a) and b) and the shifting of the graph in a DGS device in ***example*** g) founds an experimental awareness. It is nothing but: I tried it, and it worked out this way. Mathematicians know also the converse experience: Even if one has proven something in a general setting, a concrete calculation with the predicted result is not necessarily superfluous and can very well be an enhancement of one's awareness. More generally, experimental awareness can be the study of particular cases for a more general situation without worrying about the legitimacy of the particular steps supplemented by heuristic arguments.

- **Strategic awareness**

Also to find the graph of the function $g(x) = f(x+3)$ needs an appropriate strategy depending on the context. In trigonometry other ways seem appropriate than by the idea of moving the coordinate system. Especially for problem solving, at least since Poincaré, Hadamard, Polya and Schoenfeld, mathematicians are conscious of the fact that one needs more than just factual knowledge and skills. What is needed is the knowledge and the experience of steps one can take even if one does not finally know how to proceed successfully. Polya introduced for this the term *heuristics*. On the highest level, also building of a theory requires strategic knowledge: which facts will be put into axioms and which can better be proved. We see that strategic awareness goes beyond problem solving in the narrow sense.

- **Contextual awareness**

The metaphor of the two motorcyclists in ***example 3*** provides us with a context that allows getting insight into the relation between the two graphs. However, not every graph of a function may be interpreted as path-time diagram of a motorcycle. In general we can speak of contextual awareness when we attribute semantics to a mathematical topic. As well for mathematics teaching as also for professional mathematicians this type of awareness is of crucial importance. Contectual awareness constitutes *mental objects* (Freudenthal, 1991, p.19). In order to build such mental objects Freudenthal formulated the principle of *rich contexts* (p. 73).

- **Argumentative awareness**

Before proving an assertion within a theory we can argue independently of the particular theoretical framework that it must hold. For instance when we argue that it is the coordinate system that is shifted instead of the graph or when we speak of picking up the values of f, we give an argument which is not a logical one. However it serves for our argumentation. We argue by means of actions, heuristics, algorithms, and estimations etc. that help us to trust our result. We can also argue by thought experiments and or by applying metaphorical arguments.

- **Logical awareness**

Logical awareness is the awareness that ensures us of mathematical proofs and arguments. It enables us to check proofs and to distinguish heuristic arguments from logically necessary ones. But without all the other types of awareness we can hardly think logically. For example the substitution of the variable using the set-theoretic definition of the graph is a logical argument within a theoretical framework.

- **Theoretical awareness**

By theoretical awareness we understand the ability to see mathematical propositions as relative to particular theoretical frameworks, the ability to relate proposition to different frameworks and to relate frameworks to each other. Although theoretical awareness may in some sense be considered as the highest form of mathematical awareness, it becomes a kind of fata morgana without grounding on other types of mathematical awareness. In our example one could realize that the set-theoretical construction of graph is a very general construction which is used in topology or algebraic geometry to equip the graph of a function with topological or algebraic structure.

DISCUSSION

The work in our group was most inspiring. Discussions and demonstrations of various ways to link and join theories (cherry picking, organized networking, applied networking theories…) encouraged us to more problem and attention oriented developments of our approach. We are grateful for directing our attention to Regine Douady's model and realizations of the tool-object-duality and to John Mason's complex and multi-purpose work on the development and education of awareness. The latter helped us in particular to place our approach and to structure what is to be done next.

REFERENCES

Douady, R. (1997). Didactic engineering. In T. Núñez & P. Bryant (Eds.), *Learning and teaching mathematics: An International Perspective*, 373-400. East Sussex: Psychology Press.

Freudenthal, H. (1991). *Revisiting Mathematics Education, China Lectures*. Doordrecht: Kluwer Academic Publishers.

Kaenders, R.H. & Kvasz, L. (2011). Mathematisches Bewusstsein. In: Helmerich, M., Lengnink, K., Nickel, G., Rathgeb, M. (Hrsg.) Mathematik verstehen – philosophische und didaktische Perspektiven. Wiesbaden: Vieweg.

Kaenders, R.H. & Kvasz, L. (2010). *Varying Qualities of Mathematical Awareness*. Preprint – see www.mathedidaktik.uni-koeln.de/kaenders.html. Submitted to: *For the learning of mathematics.*

Kaenders, R.H. & Weiss-Pidstrygach,Y. (2010). Geometrisches Propädeutikum zur Begriffsbildung in der Analysis. In *Tagungsband der Jahrestagung der GDM 2010*. Münster: WTM-Verlag.

Kvasz, L. (2008). *Patterns of Change, Linguistic Innovations in the Development of Classical Mathematics*. Basel: Birkhäuser Verlag.

Weiss-Pidstrygach, Y. (2011). Begriffsbildung mit tätigkeitstheoretischen Methoden. *Tagungsband der Jahrestagung der GDM 2011*. Münster: WTM-Verlag.